

# REDUCED INVARIANT SETS

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*In honor of Dick Palais*

**ABSTRACT.** Let  $K$  be a compact Lie group and  $W$  a finite-dimensional real  $K$ -module. Let  $X$  be a  $K$ -stable real algebraic subset of  $W$ . Let  $\mathcal{I}(X)$  denote the ideal of  $X$  in  $\mathbb{R}[W]$  and let  $\mathcal{I}_K(X)$  be the ideal generated by  $\mathcal{I}(X)^K$ . We find necessary conditions and sufficient conditions for  $\mathcal{I}(X) = \mathcal{I}_K(X)$  and for  $\sqrt{\mathcal{I}_K(X)} = \mathcal{I}(X)$ . We consider analogous questions for actions of complex reductive groups.

## 1. INTRODUCTION

Let  $K$  be a compact Lie group, let  $W$  be a finite-dimensional real  $K$ -module and let  $X \subset W$  be  $K$ -invariant and real algebraic (the zero set of real polynomial functions on  $W$ ). Let  $\mathcal{I}(X)$  denote the ideal of  $X$  in  $\mathbb{R}[W]$ . Let  $\mathbb{R}[W]^K$  denote the  $K$ -invariants in  $\mathbb{R}[W]$  and let  $\mathcal{I}_K(X)$  be the ideal generated by  $\mathcal{I}(X)^K := \mathcal{I}(X) \cap \mathbb{R}[W]^K$ . We say that  $X$  is  *$K$ -reduced* if  $\mathcal{I}_K(X) = \mathcal{I}(X)$  and *almost  $K$ -reduced* if  $\sqrt{\mathcal{I}_K(X)} = \mathcal{I}(X)$ . Let  $Kw$  be an orbit in  $W$ . Then the *slice representation at  $w$*  is the action of the isotropy group  $K_w$  on  $N_w$ , where  $N_w$  is a  $K_w$ -complement to  $T_w(Kw)$  in  $W \simeq T_w(W)$ . An orbit  $Kw \subset W$  is *principal* (resp. *almost principal*) if the image of  $K_w$  in  $\mathrm{GL}(N_w)$  is trivial (resp. finite). We denote the principal (resp. almost principal) points of  $W$  by  $W_{\mathrm{pr}}$  (resp.  $W_{\mathrm{apr}}$ ) and we set  $X_{\mathrm{pr}} := W_{\mathrm{pr}} \cap X$  and  $X_{\mathrm{apr}} := W_{\mathrm{apr}} \cap X$ . The *strata* of  $W$  are the collections of points  $S \subset W$  whose isotropy groups are conjugate. There are finitely many strata. If  $\mathbb{R}[W]$  is a free  $R[W]^K$ -module, then we say that  $W$  is *cofree*. In the following, when we talk about one set being dense in another, we are referring to the Zariski topology.

Here are our main results:

**Theorem 1.1.** *If  $X$  is  $K$ -reduced (resp. almost  $K$ -reduced), then  $X_{\mathrm{pr}}$  (resp.  $X_{\mathrm{apr}}$ ) is dense in  $X$ , and conversely if  $W$  is cofree.*

**Theorem 1.2.** *Let  $w \in W$ . Then the orbit  $Kw$  is  $K$ -reduced (resp. almost  $K$ -reduced) if and only if  $Kw$  is principal (resp. almost principal).*

To prove the results above and to obtain further results we need to complexify. Let  $V = W \otimes_{\mathbb{R}} \mathbb{C}$  and  $G = K_{\mathbb{C}}$  be the complexifications of  $W$  and  $K$ . We have the quotient morphism  $\pi: V \rightarrow V//G$  where  $\pi$  is surjective,  $V//G$  is an affine variety and  $\pi^*\mathbb{C}[V//G] = \mathbb{C}[V]^G$ . We have the Luna strata of the quotient  $V//G$  whose inverse images in  $V$  are the strata of  $V$ . The strata of  $V$  are in 1-1 correspondence with those of  $W$  [Sch80, §5]. Let  $Y = X_{\mathbb{C}}$  be the complexification of  $X$  (the Zariski closure of  $X$  in  $V$ ). We say that  $Y$  is  *$G$ -saturated* if  $Y = \pi^{-1}(\pi(Y))$  and that  $Y$  is  *$G$ -reduced* if the ideal  $\mathcal{I}(Y)$  of  $Y$  is generated by  $\mathcal{I}(Y)^G$ . We can define  $Y_{\mathrm{apr}}$  and  $Y_{\mathrm{pr}}$  as above (see §3). If  $f_1, \dots, f_k$  are functions on a complex variety, let  $\mathcal{I}(f_1, \dots, f_k)$  denote the ideal they generate.

**Theorem 1.3.** (1)  *$X$  is almost  $K$ -reduced if and only if  $Y$  is  $G$ -saturated.*  
(2)  *$X$  is  $K$ -reduced if and only if  $Y$  is  $G$ -reduced.*  
(3)  *$X_{\mathrm{apr}}$  (resp.  $X_{\mathrm{pr}}$ ) is dense in  $X$  if and only if  $Y_{\mathrm{apr}}$  (resp.  $Y_{\mathrm{pr}}$ ) is dense in  $Y$ .*

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**Theorem 1.4.** *Assume that  $Y//G \subset V//G$  is the zero set of  $f_1, \dots, f_k$ .*

- (1) *Suppose that  $Y_{\text{apr}}$  is dense in  $Y$  and that for any stratum  $S$  of  $V$  which intersects  $Y \setminus Y_{\text{apr}}$  the codimension of  $S$  in  $V$  is at least  $k + 1$ . Then  $Y$  is  $G$ -saturated.*
- (2) *Suppose that  $Y_{\text{pr}}$  is dense in  $Y$  and that  $Y$  is  $G$ -saturated. In addition, suppose that  $\mathcal{I}(\pi(Y)) = \mathcal{I}(f_1, \dots, f_k)$  where  $Y$  has codimension  $k$  in  $V$ . Then  $Y$  is  $G$ -reduced.*

**Corollary 1.5.** *If (1) above holds, then  $X$  is almost  $K$ -reduced. If (2) holds, then  $X$  is  $K$ -reduced.*

In sections 2–4 we consider when a general  $G$ -invariant  $Y \subset V$  is  $G$ -saturated or  $G$ -reduced and we establish Theorem 1.4. In section 5 we treat the real case by complexifying. At the end of section 5 we establish Theorems 1.1, 1.2 and 1.3.

D. Ž. Đoković posed the question of identifying the  $X$  which are  $K$ -reduced. Our results give a partial answer. We thank M. Raïs for transmitting the question to us. We thank the referee for a careful reading of the manuscript, helpful suggestions and Lemma 3.3.

## 2. THE COMPLEX CASE

Let  $G$  be a complex reductive group and  $Y$  an affine algebraic set with an algebraic  $G$ -action. Dual to the inclusion  $\mathbb{C}[Y]^G \subset \mathbb{C}[Y]$  we have the quotient morphism  $\pi_Y: Y \rightarrow Y//G$ . Let  $V$  be a finite-dimensional  $G$ -module and let  $Y$  be a  $G$ -stable algebraic subset of  $V$  (the zero set of an ideal of  $\mathbb{C}[V]$ ). We shall denote  $\pi_V$  simply by  $\pi$ . Then  $\pi_Y = \pi|_Y$  and  $\pi(Y) \simeq Y//G$  is an algebraic subset of  $V//G$ . We say that  $Y$  is  $G$ -saturated if  $Y = \pi^{-1}(\pi(Y))$ . Let  $\mathcal{I}(Y)$  denote the ideal of  $Y$  in  $\mathbb{C}[V]$  and let  $\mathcal{I}_G(Y)$  denote the ideal generated by  $\mathcal{I}(Y)^G$ . We say that  $Y$  is  $G$ -reduced if  $\mathcal{I}(Y) = \mathcal{I}_G(Y)$ . The null cone  $\mathcal{N}(V)$  of  $V$  is the fiber  $\pi^{-1}(\pi(0))$ . Then  $\mathcal{N}(V)$  is (scheme theoretically) defined by the ideal  $\mathcal{I}_G(\{0\})$  so that the scheme  $\mathcal{N}(V)$  is reduced if and only if the set  $\mathcal{N}(V)$  is  $G$ -reduced, in which case we say that  $V$  is coreduced. See [KS11] for more on coreduced representations.

The points of  $V//G$  are in one-to-one correspondence with the closed  $G$ -orbits in  $V$ . The Luna strata of  $V//G$  are the sets of closed orbits whose isotropy groups are all  $G$ -conjugate. There are finitely many strata in  $V//G$ , and we consider their inverse images in  $V$  to be the strata of  $V$ . Let  $v \in V$  such that  $Gv$  is closed. Then the isotropy group  $G_v$  is reductive, and there is a  $G_v$ -stable complement  $N_v$  to  $T_v(Gv)$  in  $V \simeq T_v(V)$ . We call the action of  $G_v$  on  $N_v$  the slice representation at  $v$ .

We start with some examples.

*Example 2.1.* Let  $(V, G) = (k\mathbb{C}^n, \text{SL}_n)$ ,  $k \geq n$ . The invariants are generated by the determinants  $\det_{i_1, \dots, i_n}$  where the indices  $1 \leq i_1 < \dots < i_n \leq k$  tell us which  $n$  copies of  $\mathbb{C}^n$  to take. Then  $V$  is coreduced since  $\mathcal{N}(V)$  is the determinantal variety of  $(k \times n)$ -matrices of rank at most  $n - 1$ . See also [KS11]. All orbits outside the null cone are closed with trivial isotropy group, hence are principal.

*Example 2.2.* Let  $G \subset \text{GL}(V)$  be finite and nontrivial. Then  $\mathcal{N}(V)$  is the origin which is  $G$ -saturated but not  $G$ -reduced.

Part (2) of the proposition below follows from Serre’s criterion for reducedness [Mat80, Ch. 7]. Part (1) also follows, using the Jacobian criterion for smoothness.

**Proposition 2.3.** *Let  $Y \subset V$  be a  $G$ -saturated algebraic set.*

- (1) *If  $Y$  is  $G$ -reduced, then for every irreducible component  $Y_k$  of  $Y$  there is a point of  $Y_k$  where  $\text{rank } f = \text{codim } Y_k$ . Here  $f = (f_1, \dots, f_d): V \rightarrow \mathbb{C}^d$  and the  $f_i$  generate  $\mathcal{I}_G(Y)$ .*
- (2) *If  $\mathcal{I}_G(Y) = \mathcal{I}(f_1, \dots, f_d)$  where the  $f_i \in \mathbb{C}[V]^G$  and  $Y$  has codimension  $d$ , then  $Y$  is  $G$ -reduced if and only if the rank condition of (1) is satisfied.*

*Example 2.4.* Let  $G = \mathrm{SO}_3(\mathbb{C})$  acting as usual on  $V = 2\mathbb{C}^3$ . Then the invariants are generated by inner products  $f_{ij}$ ,  $1 \leq i \leq j \leq 2$ . Each copy of  $\mathbb{C}^3$  has a weight basis  $\{v_2, v_0, v_{-2}\}$  relative to the action of the maximal torus  $T = \mathbb{C}^*$  where  $v_j$  has weight  $j$ . The null cone  $Y := \mathcal{N}(V)$  is the  $G$ -orbit of all the vectors  $v = (\alpha v_2, \beta v_2)$  for  $\alpha, \beta \in \mathbb{C}$ . But one easily calculates that the rank of  $(f_{11}, f_{22}, f_{12}): V \rightarrow \mathbb{C}^3$  at  $v$  is at most 2 while  $Y$  has codimension 3. Thus the null cone is not  $G$ -reduced.

### 3. THE CASE WHERE $Y_{\mathrm{pr}}$ OR $Y_{\mathrm{apr}}$ IS DENSE IN $Y$

Throughout this section we assume that  $V$  is a *stable* representation of  $G$ , i.e., there is a nonempty open subset of closed orbits. This is always the case when  $(V, G) = (W_{\mathbb{C}}, K_{\mathbb{C}})$  is a complexification ([Lun72] or [Sch80, Cor. 5.9]). Let  $Gv$  be a closed orbit. We say that  $Gv$  is *principal* if the slice representation  $(N_v, G_v)$  is a trivial representation and that  $Gv$  is *almost principal* if  $G_v \rightarrow \mathrm{GL}(N_v)$  has finite image. We denote the principal (resp. almost principal) points of  $V$  by  $V_{\mathrm{pr}}$  (resp.  $V_{\mathrm{apr}}$ ). If  $Y \subset V$  is  $G$ -stable, we set  $Y_{\mathrm{pr}} = Y \cap V_{\mathrm{pr}}$  and  $Y_{\mathrm{apr}} = Y \cap V_{\mathrm{apr}}$ . Both  $Y_{\mathrm{apr}}$  and  $Y_{\mathrm{pr}}$  are open in  $Y$ . In general, the fiber of  $\pi$  through a closed orbit  $Gv \subset V$  is  $G \times^{G_v} \mathcal{N}(N_v)$  (the  $G$ -fiber bundle with fiber  $\mathcal{N}(N_v)$  associated to the  $G_v$ -principal bundle  $G \rightarrow G/G_v$ ). Thus the fiber is set-theoretically the orbit if and only if  $\mathcal{N}(N_v)$  is a point. This happens if and only if the image  $G_v \rightarrow \mathrm{GL}(N_v)$  is finite, i.e.,  $v \in V_{\mathrm{apr}}$ . Hence  $Y_{\mathrm{apr}}$  is always  $G$ -saturated. Similarly, the fiber is scheme-theoretically the orbit if and only if  $\mathcal{N}(N_v)$  is schematically a point which is equivalent to  $G_v$  acting trivially on  $N_v$ , i.e., we have  $v \in V_{\mathrm{pr}}$ . Hence  $Y_{\mathrm{pr}}$  is always  $G$ -reduced. To sum up we have

**Proposition 3.1.** *Let  $Gv$  be a closed orbit and let  $Y \subset V$  be a  $G$ -stable algebraic set.*

- (1) *If  $Y = Y_{\mathrm{apr}}$ , then  $Y$  is  $G$ -saturated. In particular,  $Gv$  is  $G$ -saturated if and only if it is almost principal.*
- (2) *If  $Y = Y_{\mathrm{pr}}$ , then  $Y$  is  $G$ -reduced. In particular,  $Gv$  is  $G$ -reduced if and only if it is principal.*
- (3) *The fiber  $\pi^{-1}(\pi(v))$  is  $G$ -reduced if and only if the slice representation  $(N_v, G_v)$  is coreduced.*

**Corollary 3.2.** *If the isotropy groups of  $G$  acting on  $Y$  are all finite, then  $Y$  is  $G$ -saturated and if  $G$  acts freely on  $Y$ , then  $Y$  is  $G$ -reduced.*

Of course, it is possible that  $Y$  is  $G$ -saturated (resp.  $G$ -reduced) even if  $Y_{\mathrm{apr}}$  (resp.  $Y_{\mathrm{pr}}$ ) is empty. But in the case of a complexification  $Y = X_{\mathbb{C}}$  it is necessary for  $G$ -saturation (resp.  $G$ -reducedness) that  $Y_{\mathrm{apr}}$  (resp.  $Y_{\mathrm{pr}}$ ) is dense in  $Y$  (see §5). We consider the case that  $Y_{\mathrm{apr}}$  or  $Y_{\mathrm{pr}}$  is not dense in  $Y$  in the next section.

Unfortunately, we do not have the analogues of Proposition 3.1(1) and (2) for  $X$ . See Example 5.3 below.

Recall that  $V$  is *cofree* if  $\mathbb{C}[V]$  is a free  $\mathbb{C}[V]^G$ -module. Equivalently,  $\pi: V \rightarrow V//G$  is flat, or  $\mathbb{C}[V]^G$  is a regular ring and the codimension of  $\mathcal{N}(V)$  is  $\dim \mathbb{C}[V]^G$  [Sch80, Proposition 17.29].

We owe the following lemma to the referee.

**Lemma 3.3.** *Let  $V$  be a cofree  $G$ -module and let  $U \subset V//G$  be locally closed.*

- (1) *We have  $\overline{\pi^{-1}(U)} = \pi^{-1}(\overline{U})$ .*
- (2) *If  $\pi^{-1}(U)$  is reduced, then so is  $\pi^{-1}(\overline{U})$ .*

*Proof.* For (1) set  $Z := \overline{\pi^{-1}(U)}$ . Then  $\pi(Z)$  is closed [Kra84, II.3.2], hence  $\pi(Z) = \overline{U}$ . Since  $\pi$  is flat, so is  $\pi^{-1}(\overline{U}) \rightarrow \overline{U}$ . Set  $S := \pi^{-1}(\overline{U}) \setminus Z$ . Then  $S$  is open, hence  $\pi(S)$  is open in  $\overline{U}$  (by flatness). By construction,  $\pi(S)$  does not meet  $U$ , hence we must have  $S = \emptyset$ , establishing (1).

For (2) we can assume that  $U = \overline{U}_f = \{u \in \overline{U} \mid f(u) \neq 0\}$  for  $f \in \mathbb{C}[\overline{U}]$ . Set  $Z := \pi^{-1}(\overline{U})$ , the schematic inverse image of  $\overline{U}$ . Since  $\mathbb{C}[\overline{U}] \rightarrow \mathbb{C}[U] = \mathbb{C}[\overline{U}]_f$  is injective and  $\mathbb{C}[Z]$  is flat over  $\mathbb{C}[\overline{U}]$ , it follows that  $\mathbb{C}[Z] \rightarrow \mathbb{C}[Z]_f = \mathbb{C}[\pi^{-1}(U)]$  is also injective. Since the latter ring is reduced, so is  $\mathbb{C}[Z]$  and we have established (2).  $\square$

**Corollary 3.4.** *Suppose that  $(V, G)$  is cofree and that  $Y \subset V$  is a  $G$ -stable algebraic set such that  $Y_{\text{apr}}$  is dense in  $Y$ . Then  $Y$  is  $G$ -saturated.*

*Example 3.5.* Let  $(V, G) = (4\mathbb{C}^2, \text{SL}_2)$  and let  $Y = 2\mathbb{C}^2 \times \{0\}$ . Then  $Y_{\text{pr}} = Y_{\text{apr}}$  is dense in  $Y$  (it is the set of linearly independent vectors in  $Y$ ) but  $Y$  is not  $G$ -saturated since it does not contain the null cone. The  $G$ -module  $V$  is not cofree, so we don't contradict Corollary 3.4. Note that this example is the complexification of the case where  $X = \mathbb{C}^2 \times \{0\} \subset W := \mathbb{C}^2 \oplus \mathbb{C}^2$  and  $K = \text{SU}(2, \mathbb{C})$ . Thus  $X_{\text{pr}} = X_{\text{apr}}$  is dense in  $X$  but  $X$  is not almost  $K$ -reduced. (We use Theorem 1.3.) This shows that cofreeness is also necessary in Theorem 1.1.

**Theorem 3.6.** *Suppose that  $Y \subset V$  is  $G$ -stable such that*

- (1)  $Y_{\text{apr}}$  is dense in  $Y$ .
- (2)  $Y//G \subset V//G$  is the zero set of  $f_1, \dots, f_k$  where the minimal codimension of a non almost principal stratum of  $V$  which intersects  $Y$  is at least  $k + 1$ .

*Then  $Y$  is  $G$ -saturated.*

*Proof.* Let  $\tilde{Y}$  denote  $\pi^{-1}(\pi(Y))$ . Then each irreducible component of  $\tilde{Y}$  has codimension less than or equal to  $k$ . Let  $S$  be a non almost principal stratum of  $V$  which intersects  $Y$ . Then  $S \cap \tilde{Y}$  is nowhere dense in  $\tilde{Y}$ . Thus  $\tilde{Y}_{\text{apr}}$  is dense in  $\tilde{Y}$ . Now  $\tilde{Y}_{\text{apr}}$  and  $Y_{\text{apr}}$  have the same image in  $Y//G$ . Hence  $Y_{\text{apr}} = \tilde{Y}_{\text{apr}}$  and  $Y = \tilde{Y}$  is saturated.  $\square$

*Example 3.7.* Let  $(V, G) = (k\mathbb{C}^2, \text{SL}_2)$ ,  $k \geq 2$ . The codimension of the null cone is  $k - 1$  and the subset  $Y$  where the first copy of  $\mathbb{C}^2$  is zero is not saturated, but corresponds to the subset of  $V//G$  where the determinant invariants  $\det_{12}, \dots, \det_{1k}$  vanish (see Example 2.1). Thus the codimension condition in Theorem 3.6(2) is sharp.

Here is an example that is a complexification.

*Example 3.8.* Let  $(V, G) = (2\mathbb{C}^2, \text{SO}_2(\mathbb{C}))$  and let  $Y = \mathbb{C}^2 \times \{0\} \cup \{0\} \times \mathbb{C}^2$ . Then  $Y_{\text{apr}}$  is dense in  $Y$  since any point not in  $\mathcal{N}(V)$  is on a principal orbit and  $\mathcal{N}(V)$  is nowhere dense in  $Y$ . However,  $Y$  is not  $G$ -saturated since it does not contain  $\mathcal{N}(V)$ . Note that  $\mathcal{I}(Y//G)$  is generated by  $\det$  (the determinant),  $f_{12}$  and  $f_{11}f_{22}$  where the  $f_{ij}$  are the inner product invariants. Since  $\det^2 = f_{11}f_{22} - f_{12}^2$ ,  $\mathcal{I}(Y//G)$  is the radical of the ideal generated by  $f_{12}$  and  $f_{11}f_{22}$ . The null cone has codimension 2. Again this shows that the codimension condition in Theorem 3.6 is sharp.

We now have the following corollary of Lemma 3.3

**Corollary 3.9.** *Suppose that  $(V, G)$  is cofree and that  $Y \subset V$  is  $G$ -stable such that  $Y_{\text{pr}}$  is dense in  $Y$ . Then  $Y$  is  $G$ -reduced.*

*Remark 3.10.* For  $Y$  to be  $G$ -reduced, it is not sufficient that every slice representation of  $V$  is coreduced. (This is the same as saying that every fiber of  $\pi: V \rightarrow V//G$  is reduced.) Just consider Example 3.5 again. Here  $Y_{\text{pr}}$  is dense in  $Y$  but  $Y$  is not  $G$ -saturated, let alone  $G$ -reduced.

**Theorem 3.11.** *Let  $V$  be a  $G$ -module and let  $Y \subset V$  be  $G$ -saturated such that  $Y_{\text{pr}}$  is dense in  $Y$ . Suppose that  $\pi(Y) \subset V//G$  is the zero set of  $f_1, \dots, f_k$  where the codimension of  $Y$  is  $k$ . Then  $Y$  is  $G$ -reduced.*

*Proof.* The rank of the differential of  $f = (f_1, \dots, f_d): V \rightarrow \mathbb{C}^d$  is maximal at a point of each irreducible component of  $Y$  since  $Y$  is reduced at all points of  $Y_{\text{pr}}$ . Thus we can apply Serre's criterion (Proposition 2.3).  $\square$

*Example 3.12.* Let  $(V, G) = (4\mathbb{C}^2, \text{SL}_2(\mathbb{C}))$  and let  $Y$  be the zero set of two of the determinant invariants  $\det_{ij}$ . Then  $Y_{\text{pr}}$  is dense in  $Y$  since the only non-principal stratum is  $\mathcal{N}(V)$  which has codimension 3 while  $Y$  has codimension 2. By Theorem 3.11,  $Y$  is  $G$ -reduced.

#### 4. THE CASE WHERE $Y_{\text{pr}}$ OR $Y_{\text{apr}}$ IS NOT DENSE IN $Y$

We can say something in the case that  $Y_{\text{apr}}$  or  $Y_{\text{pr}}$  is not dense in  $Y$ . We are certainly in this case if  $V$  is not stable, since then  $V_{\text{pr}}$  and  $V_{\text{apr}}$  are empty. Let  $v \in Y$  such that  $Gv$  is closed. Let  $(N_v, G_v)$  be the slice representation and  $S$  the corresponding stratum of  $V$ . We say that  $(N_v, G_v)$  is a *generic slice representation for  $Y$*  if  $S \cap Y$  is dense in an irreducible component of  $Y$ . We also say that  $S$  is *generic for  $Y$* .

**Proposition 4.1.** *Let  $(N_v, G_v)$  be a generic slice representation of  $Y$  corresponding to the stratum  $S$  of  $V$ . If  $Y$  is  $G$ -saturated, then  $Y \cap S = \pi^{-1}(\pi(Y \cap S))$ . If  $Y$  is  $G$ -reduced, then  $(N_v, G_v)$  is coreduced.*

*Proof.* If  $Y$  is  $G$ -saturated, then we obviously must have that  $Y \cap S = \pi^{-1}(\pi(Y \cap S))$ . Let  $Z$  denote  $\pi(S)$ . Then  $\pi^{-1}(Z) \rightarrow Z$  is a fiber bundle with fiber  $G \times^{G_v} \mathcal{N}(N_v)$ . If  $Y$  is  $G$ -reduced, then the bundle is reduced, hence  $(N_v, G_v)$  is coreduced.  $\square$

Let  $S$  be a stratum of  $V$ . We say that  $Y$  is  *$S$ -saturated* if  $Y \cap S = \pi^{-1}(\pi(Y \cap S))$ . We say that  $Y$  is  *$S$ -reduced* if  $Y$  is  $S$ -saturated and the slice representation  $(N_v, G_v)$  associated to  $S$  is coreduced. Corresponding to Corollaries 3.4 and 3.9 and Theorems 3.6 and 3.11 we have the following result whose proof we leave to the reader.

**Theorem 4.2.** *Let  $Y \subset V$  be a  $G$ -stable algebraic set.*

- (1) *If  $V$  is cofree and  $Y$  is  $S$ -saturated for every stratum  $S$  which is generic for  $Y$ , then  $Y$  is  $G$ -saturated.*
- (2) *If  $V$  is cofree and  $Y$  is  $S$ -reduced for every stratum  $S$  which is generic for  $Y$ , then  $Y$  is  $G$ -reduced.*
- (3) *Suppose that  $Y$  is  $S$ -saturated for every every generic stratum  $S$  of  $Y$ . Further assume that the minimal codimension of the strata of  $V$  which intersect  $Y$  but are not generic for  $Y$  is greater than  $k$  and that  $Y//G$  is the zero set of  $f_1, \dots, f_k$ . Then  $Y$  is  $G$ -saturated.*
- (4) *Suppose that  $Y$  is  $G$ -saturated and that the ideal of  $\pi(Y) \subset V//G$  is generated by  $f_1, \dots, f_k$  where the codimension of  $Y$  in  $V$  is  $k$ . Also assume that  $Y$  is  $S$ -reduced for every generic stratum  $S$  of  $Y$ . Then  $Y$  is  $G$ -reduced.*

#### 5. THE REAL CASE

Let  $W$  be a real  $K$ -module where  $K$  is compact. Let  $X \subset W$  be real algebraic and  $K$ -stable. Now  $K$  is naturally a real algebraic group and the action on  $W$  is real algebraic. Moreover, every orbit of  $K$  in  $W$  is a real algebraic set [Sch01]. Let  $Y := X_{\mathbb{C}}$  denote the complexification of  $X$  inside  $V := W \otimes_{\mathbb{R}} \mathbb{C}$  and let  $G$  denote the complexification  $K_{\mathbb{C}}$  of  $K$ . Then  $G$  is reductive and  $V$  is a stable  $G$ -module ([Lun72] or [Sch80, Cor. 5.9]). We say that a slice representation  $(N_w, K_w)$  is a *generic slice representation for  $X$*  if  $w \in X$  and the corresponding stratum contains a nonempty open subset of  $X$ . Equivalently, the complexification of  $(N_w, K_w)$  is generic for  $Y$ .

**Proposition 5.1.** (1)  *$X$  is almost  $K$ -reduced if and only if  $Y$  is  $G$ -saturated.*

(2)  *$X$  is  $K$ -reduced if and only if  $Y$  is  $G$ -reduced.*



- (3) The set  $X_{\text{apr}}$  (resp.  $X_{\text{pr}}$ ) is dense in  $X$  if and only if the set  $Y_{\text{apr}}$  (resp.  $Y_{\text{pr}}$ ) is dense in  $Y$ .
- (4)  $X$  is almost  $K$ -reduced implies that  $X_{\text{apr}}$  is dense in  $X$ .
- (5)  $X$  is  $K$ -reduced implies that  $X_{\text{pr}}$  is dense in  $X$ .

*Proof.* The ideal of  $Y$  is  $\mathcal{I}(X) \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{R}[W] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[V]$  and  $\mathcal{I}_K(X) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{I}_G(Y)$ . Thus  $\mathcal{I}(Y) = \mathcal{I}_G(Y)$  if and only if  $\mathcal{I}(X) = \mathcal{I}_K(X)$ , and  $\mathcal{I}(Y) = \sqrt{\mathcal{I}_G(Y)}$  if and only if  $\mathcal{I}(X) = \sqrt{\mathcal{I}_K(X)}$ . Hence we have (1) and (2). For (3), note that  $X_{\text{apr}}$  is open in  $X$  and that  $Y_{\text{apr}}$  is open in  $Y$ . If a stratum  $S$  of  $W$  is dense in an irreducible component of  $X$ , then the corresponding stratum  $S_{\mathbb{C}}$  of  $V$  is dense in an irreducible component of  $Y$ . Thus if  $X_{\text{apr}}$  is not dense in  $X$ , then  $Y_{\text{apr}}$  is not dense in  $Y$ . Clearly, if  $X_{\text{apr}}$  is dense in  $X$ ,  $Y_{\text{apr}} \supset X_{\text{apr}}$  is dense in  $Y$ . The argument for  $X_{\text{pr}}$  and  $Y_{\text{pr}}$  is similar, hence we have (3). Now suppose that  $X$  is almost  $K$ -reduced. Then for  $S$  a generic stratum of  $X$  and  $x \in S \cap X$ , the complexification  $Gx \simeq G/G_x$  of  $Kx$  is Zariski dense in the fiber  $G \times^{G_x} \mathcal{N}(W_x \otimes_{\mathbb{R}} \mathbb{C})$  where  $G_x = (K_x)_{\mathbb{C}}$ . Thus  $\mathcal{N}(W_x \otimes_{\mathbb{R}} \mathbb{C})$  is a point, i.e., the stratum consists of almost principal orbits. Hence we have (4), and (5) is proved similarly.  $\square$

**Corollary 5.2.** *Let  $X = Kw$  be an orbit. Then  $X$  is almost  $K$ -reduced if and only if  $Kw$  is almost principal and  $X$  is  $K$ -reduced if and only if  $Kw$  is principal.*

Unfortunately, it is not true that  $X = X_{\text{pr}}$  (or  $X = X_{\text{apr}}$ ) implies the same equality for  $Y$ .

*Example 5.3.* Let  $K = \text{SU}_2(\mathbb{C})$  and  $W = 2\mathbb{C}^2 \oplus \mathbb{R}$  where  $K$  acts as usual on the copies of  $\mathbb{C}^2$  and trivially on  $\mathbb{R}$ . We consider  $W$  to be  $2\mathbb{H} \oplus \mathbb{R}$  where  $\mathbb{H}$  denotes the quaternions. Then  $K \simeq S^3$ , the unit quaternions, and the action on  $2\mathbb{H}$  is given by  $k(p, q) = (kp, kq)$ ,  $p, q \in \mathbb{H}$ ,  $k \in S^3$ . Let  $p \mapsto \bar{p}$  denote the usual conjugation of quaternions. The invariants of  $K$  acting on  $2\mathbb{H}$  are generated by  $(p, q) \mapsto (\bar{p}p, \bar{q}q, \bar{q}p)$  where the first two invariants lie in  $\mathbb{R}$  and the last in  $\mathbb{H}$ . Let  $\alpha$  and  $\beta$  denote the first two invariants and let  $\gamma$  be the real part of  $\bar{q}p$ . Let  $\delta, \epsilon$  and  $\zeta$  be the invariants which are the  $i, j$  and  $k$  components of  $\bar{q}p$ , respectively. Then there are certainly points in  $2\mathbb{H}$  where  $\delta, \epsilon$  and  $\zeta$  vanish and where  $\alpha = \beta = \gamma$  is any positive real number. Let  $x$  be a coordinate on the copy of  $\mathbb{R}$  in  $W$  and let  $X$  be the subset of  $W$  defined by  $\delta = \epsilon = \zeta = 0$ ,  $\alpha = \beta = \gamma$  and  $(\alpha - 1)^2 + x^2 = 1/2$ . Then  $\alpha$  never vanishes on  $X$  which implies that the isotropy group at the corresponding point of  $W$  is trivial, so we have that  $X = X_{\text{pr}}$ . The quotient  $X/K$  is a smooth curve, hence  $X$  is smooth of dimension 4. The complexification  $Y$  of  $X$  also has dimension four and contains some of the points  $(s, t, \pm\sqrt{-3/4})$  where  $(s, t)$  lies in the null cone of  $2\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq 4\mathbb{C}^2$  for the action of  $K_{\mathbb{C}} \simeq \text{SL}_2(\mathbb{C})$ . But this null cone has dimension 5. Hence  $Y$  is not  $G$ -saturated, let alone  $G$ -reduced, and  $Y \neq Y_{\text{apr}}$ . Moreover,  $X$  is neither  $K$ -reduced nor almost  $K$ -reduced.

Now we recover the theorems of the introduction. Theorem 1.2 is just Corollary 5.2. Theorem 1.3 is a consequence of Proposition 5.1 and Theorem 1.4 follows from Theorems 3.6 and 3.11.

*Proof of Theorem 1.1.* Suppose that  $X$  is  $K$ -reduced. Then Proposition 5.1 shows that  $X_{\text{pr}}$  is dense in  $X$ . Conversely, if  $(W, K)$  is cofree (equivalently,  $(V, G)$  is cofree) and  $X_{\text{pr}}$  is dense in  $X$ , then  $Y_{\text{pr}}$  is dense in  $Y$  by Proposition 5.1 and  $Y$  is  $G$ -reduced by Corollary 3.9. Hence  $X$  is  $K$ -reduced. The proof in the almost  $K$ -reduced case is similar.  $\square$

## REFERENCES

- [Kra84] Hanspeter Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, D1, Friedr. Vieweg & Sohn, Braunschweig, 1984.
- [KS11] Hanspeter Kraft and Gerald W. Schwarz, *Reduced null cones*, to appear.
- [Lun72] Domingo Luna, *Sur les orbites fermées des groupes algébriques réductifs*, Invent. Math. **16** (1972), 1–5.
- [Mat80] Hideyuki Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

- [Sch80] Gerald W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 51, 37–135.
- [Sch01] ———, *Algebraic quotients of compact group actions*, J. Algebra **244** (2001), no. 2, 365–378.

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